Min. Error Disc. of Linearly Independent Pure States

Analytic Properties of Optimal POVM

Tanmay Singal[1]   Sibasish Ghosh[1]

[1]Optics and Quantum Information Group
Institute of Mathematical Sciences
Chennai

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The Ensemble

- $\{p_i > 0, \rho_i\}_{i=1}^{m}$ $\quad \rho_i \in \mathcal{B}(\mathcal{H})$ where $\dim\mathcal{H} = n$

The m POVM

- $\{\Pi_i\}_{i=1}^{m}$ $\quad \Pi_i \geq 0$ $\quad \sum_{i=1}^{m} \Pi_i = \mathbb{I}$

- Avg. Probability of Success: $P_s = \sum_{i=1}^{m} p_i Tr(\rho_i \Pi_i)$
- Avg. Probability of Error: $P_e = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} p_i Tr(\rho_i \Pi_j)$

- $P_s^{\max} = \text{Max}_{\{\Pi_i\}_{i=1}^{m}} P_s = 1 - P_e^{\min}$
- OBJECTIVE: Find Optimal m-POVM!

Necessary And Sufficient Conditions

- Necessary Condition: $\Pi_j(p_j\rho_j - p_i\rho_i)\Pi_i = 0, \forall 1 \leq i, j \leq m$ (Given by Holevo)
- SDP Necessary and Sufficient Conditions:

- $\text{Min}_{Z=Z^\dagger}\text{Tr}(Z) \ni Z \geq p_i\rho_i \, \forall 1 \leq i \leq m$
- Complementary Slackness Conditions (given by Yuen): $(Z - p_i\rho_i)\Pi_i = \Pi_i(Z - p_i\rho_i) = 0, \, \forall 1 \leq i \leq m$

Some Important Ensembles Solved So Far

- When Supp(ρ_i)$\perp$ Supp(ρ_j) $\forall\, i, j$
  Π_i = Projector on Supp(ρ_i) $\forall\, i$
- Any ensemble of two states (by Helstrom)
- When $\sum_i p_i \rho_i = \frac{1}{m}\mathcal{I}$
- Equiprobable ensemble when ρ_i lie on the orbit of some unitary: $\rho_{i+1} = U\ \rho_i\ U^\dagger$
- For three qubits (recently)

Problem We Look At

- The problem we look at is MED when $\rho_i \rightarrow |\psi_i\rangle\langle\psi_i|$ and $\{ \psi_i \}$ are linearly independent.
- In this case $\dim(\mathcal{H}) = m$
- Kennedy showed that
  $\Pi_i \rightarrow |v_i\rangle\langle v_i|$, where $\langle v_i|v_j\rangle = \delta_{i,j} \; \forall \; 1 \leq i, j \leq m$.
  i.e. Optimal m-POVM is rank-one projective measurement
- Case when m = 2 has been solved, but not for m = 3.

A Special Representation

- Ensemble: $\{p_i, \, |\psi_i\rangle\langle\psi)i|\}_{i=1}^m$ where $|\psi_i\rangle$ are LI.
- Define: $|\widetilde{\psi}_i\rangle \; = \; \sqrt{p_i}|\psi_i\rangle$
- $|\widetilde{\psi}_i\rangle$ are LI $\Rightarrow \; \exists \, \{|\widetilde{u}_i\rangle\} \; \ni \; \langle\widetilde{\psi}_i|\widetilde{u}_j\rangle \; = \; \delta_{i,j}$
- Let G be gram matrix of $\{ \, |\psi_i\rangle\}$ i.e. $G_{ij} = \langle\widetilde{\psi}_i|\widetilde{\psi}_j\rangle$ which implies and is implied by $G^{-1}_{ij} = \langle\widetilde{u}_i|\widetilde{u}_j\rangle$
- Let $\{|v_i\rangle\}$ be any ONB. Then $|v_i\rangle = \sum_{j=1}^m (\mathrm{G}^{\frac{1}{2}} U)_{j,i}|\widetilde{u}_j\rangle$ where U is some unitary.

- Holevo's Necessary Condition:

$$\Rightarrow \langle v_j|(|\widetilde{\psi}_j\rangle\langle\widetilde{\psi}_j| - |\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|)|v_i\rangle = 0, \ \forall\, 1 \leq \ i,j \ \leq m$$

$$\Rightarrow \ (\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{ii}(\mathrm{G}^{\frac{1}{2}}\widetilde{U})^*_{ij} \ = \ (\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{ji}(\mathrm{G}^{\frac{1}{2}}\widetilde{U})^*_{jj}\ \forall\ 1 \leq i,j \leq m$$

- Add a phase $\longrightarrow |v_i\rangle \longrightarrow e^{i\phi_i}|v_i\rangle$ so that $(\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{i,i} \geq 0$
- Let $\mathrm{D} = \mathrm{Diag}((\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{11}), (\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{22}), \cdots, (\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{mm})$
- Then the matix $\mathrm{G}^{\frac{1}{2}}\widetilde{U}D^{-1}$ is hermitian.

$$\mathrm{G}^{\frac{1}{2}}\widetilde{U}D^{-1} = \begin{pmatrix} 1 & \frac{(\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{1,2}}{(\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{2,2}} & \cdots \\ \frac{(\mathrm{G}^{\frac{1}{2}}\widetilde{U})_{2,1}}{(\mathrm{G}^{\frac{1}{1}}\widetilde{U})_{1,1}} & 1 & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix}$$

Consider the equation:

$$D^{-1}\widetilde{U}^{\dagger}G^{\frac{1}{2}}\ G^{-1}\ G^{\frac{1}{2}}\widetilde{U}D^{-1} = D^{-2}$$

- Note that the RHS is diagonal.
- Now, let $m=3$ and the states $|\psi_i\rangle$ be real $\Rightarrow$ everything is real (gram matrix, optimal POVM, $|\widetilde{u_i}\rangle$ states etc). So we can work in the real domain without having to worry about complex numbers.
- The matrix equation upstairs is then given by[1]:

$$\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix} \begin{pmatrix} (G^{-1})_{11} & (G^{-1})_{12} & (G^{-1})_{13} \\ (G^{-1})_{21} & (G^{-1})_{22} & (G^{-1})_{23} \\ (G^{-1})_{31} & (G^{-1})_{32} & (G^{-1})_{33} \end{pmatrix} \begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix} = \begin{pmatrix} (D_{11})^{-2} & 0 & 0 \\ 0 & (D_{22})^{-2} & 0 \\ 0 & 0 & (D_{33})^{-2} \end{pmatrix}$$

[1]Note again that the elements $(G^{-1})_{ij}$ are known and that $\alpha, \beta, \gamma, D_{ii}$ are unknowns.

We get 6 equations in the matrix elements:

$$\alpha^2(G^{-1})_{12} + \alpha((G^{-1})_{11} + (G^{-1})_{22} + (G^{-1})_{13}\beta + (G^{-1})_{23}\gamma) + (G^{-1})_{33}\beta\gamma + (G^{-1})_{23}\beta + (G^{-1})_{13}\gamma + (G^{-1})_{12} = 0$$

$$\beta^2(G^{-1})_{13} + \beta((G^{-1})_{11} + (G^{-1})_{33} + (G^{-1})_{23}\gamma + (G^{-1})_{12}\alpha) + (G^{-1})_{22}\alpha\gamma + (G^{-1})_{12}\gamma + (G^{-1})_{23}\alpha + (G^{-1})_{13} = 0$$

$$\gamma^2(G^{-1})_{23} + \gamma((G^{-1})_{22} + (G^{-1})_{33} + (G^{-1})_{13}\beta + (G^{-1})_{12}\alpha) + (G^{-1})_{11}\alpha\beta + (G^{-1})_{12}\beta + (G^{-1})_{13}\alpha + (G^{-1})_{23} = 0$$

The above are quadraitc in α, β, γ. Typically this set has 8 different solutions of which only one corr - esponds to the optimal POVM. Some solutions are complex and hence discarded. The remaining solutions correspond to projective measurements where P_s is stationary in the space of projective measum - rement.

$$(G^{-1})_{22}\alpha^2 + (G^{-1})_{33}\beta^2 + 2\alpha\beta(G^{-1})_{23} + 2\alpha(G^{-1})_{12} + 2\beta(G^{-1})_{13} + (G^{-1})_{11} = (D_{11}^{-2})$$

$$(G^{-1})_{11}\alpha^2 + (G^{-1})_{33}\gamma^2 + 2\alpha\gamma(G^{-1})_{13} + 2\alpha(G^{-1})_{12} + 2\gamma(G^{-1})_{23} + (G^{-1})_{22} = (D_{22}^{-2})$$

$$(G^{-1})_{11}\beta^2 + (G^{-1})_{22}\gamma^2 + 2\beta\gamma(G^{-1})_{12} + 2\beta(G^{-1})_{13} + 2\gamma(G^{-1})_{23} + (G^{-1})_{33} = (D_{33}^{-2})$$

We solve for $(\alpha,\ \beta,\ \gamma)$ from the three equations and substitute in the three equations below to obtain t he values for D_{ii}s.
A closed form solution for the equations above is very difficult to obtain. It's worth mentioning that solving the m=3 case in other methods yields polynomial equations (but often in greater number of u

nknowns). This is due to the fact there are multiple stationary points in the space of projective measurements.

$$DG^{\frac{1}{2}}\widetilde{U} = \begin{pmatrix} (G^{\frac{1}{2}}\widetilde{U})^2_{11} & (G^{\frac{1}{2}}\widetilde{U})_{11}(G^{\frac{1}{2}}\widetilde{U})_{12} & \cdots \\ (G^{\frac{1}{2}}\widetilde{U})_{21}(G^{\frac{1}{2}}\widetilde{U})_{22} & (G^{\frac{1}{2}}\widetilde{U})^2_{11} & \cdots \\ \vdots & \ddots & \cdots \end{pmatrix}$$

Hence $DG^{\frac{1}{2}}\widetilde{U}$ is hermitian.
Now

$$(\widetilde{U}^{\dagger}G^{\frac{1}{2}}D)(D^{-1}G^{-1}D^{-1})(DG^{\frac{1}{2}}\widetilde{U}) = \mathbb{I}$$

$DG^{\frac{1}{2}}\widetilde{U}$ is a hermitian square root of DGD. Note that until now we have only used Holevo's necessary conditions. What Carlos Mochos[2] and Belavkin[3], proved, proves that $DG^{\frac{1}{2}}\widetilde{U}$ is a positive square root of DGD.

[2] Phys. Rev. A 73, 032338 (2006)

[3] P. Belavkin, âOptimal multiple quantum statistical hypothesis testing.â Stochastics 1, 315-345 (1975).

From previous slide we have the equation:

$$(DG^{\frac{1}{2}}\widetilde{U})^2 - DGD = 0$$

Implicit Function Theorem

Implicit Function Theorem: Let $\{y_i\}_{i=1}^N$ be N functions (real or complex) of the independent variables - $\{t, f_i\}$ where the variables $\{f_i\}_i$, which are N in number too. Let (τ, ϕ_i) be a point such that $y_i(\tau, \phi_i) = 0 \ \forall \ 1 \le i \le N$. If the Jacobian matrix $J_{i,j} = \frac{\partial y_i}{\partial f_j}$ is invertible at (τ, ϕ_i) then there exists some open neighbourhood of τ, T for which there exists open neighbourhoods S_i around ϕ_i such that $f_i : T \longrightarrow S_i$ can be defined, so that $y_i(t, f_i(t)) = 0 \ \forall \ 1 \le i \le N$. That is

$\{(t, f_i) \in T \times S \,|\, y(t, f_i) = 0\} = \{(t, f_i(t)) \,|\, t \in T \text{ and } y(t, f_i(t)) = 0\}$ where $S = S_1 \times S_2 \times \cdots \times S_N$.

Analytic Implicit Function Theorem: Furthermore if y_i is an analytic function in the variables f_i then the implicit dependence of f_i on t will be analytic too.

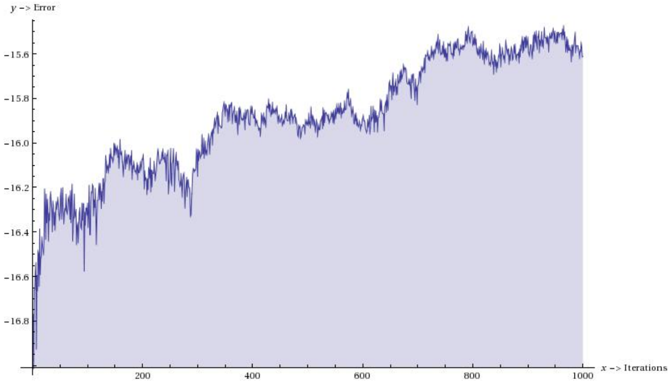
- Here $a_i = \sqrt{(\sqrt{G}\tilde{U})_{ii}} = D_{ii}$, $f_{ij} = (\sqrt{G}\tilde{U})_{ij}$, $y_{ij} = ((D\sqrt{G}\tilde{U})^2 - DGD)_{ij}$, y_{ij} is analytic in f_{ij} and a_i.
- Consider a trajectory of gram matrix: G(t)
  Objective: That a_i, f_{ij} depend implicitly on t so that $y_{ij} = 0 \ \forall t \in [0, 1]$.
- Unable to prove that the Jacobian is non-singular at every point. But we know the functions $f_{ij}(t)$ and $a_i(t)$ exist.
- The uniqueness of the optimal POVM for an ensemble $\Rightarrow$ Opt. POVM varies continuously as function of ensemble.
- Thus the function exists and is continuous. Since y_{ij} is polynomial in the variables f_{ij} and a_i, $f_{ij}(t)$ and $a_i(t)$ are analytic in t.

Dragging the Solution from One Point to Another

- Using implicit function theorem we get differential equations:
  $\frac{dy_{ij}}{dt} = \zeta_{ij}(t; a_k(t), f_{kl}(t), \frac{da_k(t)}{dt}, \frac{df_{kl}(t)}{dt}) = 0 \ \forall\ 1 \leq i,j \leq m$
  These are non-linear coupled ordinary differential equations.
- Let the starting point (initial conditions) the equiprobable orthogonal ensemble. Gram matrix corresponding to this ensemble is: $G(0) = \frac{1}{m}\mathbb{I}$
  The trajectory we employ is linear in t: $G(t) = (1-t)\frac{1}{m}\mathbb{I} + t\ G$
  where G is the gram matrix of the system which we want to solve MED for.
- We use RK4 to solve this system of coupld differential equations
  - Interval: $h = 10^{-3}$ No. of iterations: 1000
  - RK4 Local Truncation Error (expected): $\mathcal{O}(h^5) \sim -15$
    RK4,Total Accumulated Error(expected): $\mathcal{O}(h^4) \sim -12$
  - RK4 Avg. Local Truncation Error(Randomly generated for $m=5$): $\mathcal{O}(-16)$
  - RK4 Avg. Total Accumulated Error(Randomly generated for $m=5$): $\mathcal{O}(-15)$

As an illustration,

$$G = \begin{pmatrix} 0.3 & \sqrt{0.06}(0.2+i0.1) & \sqrt{0.06}(0.1) & \sqrt{0.045}(0.1) & \sqrt{0.045}(0.1) \\ \sqrt{0.06}(0.2-i0.1) & 0.2 & 0.06 & \sqrt{0.03}(0.2+i0.2) & \sqrt{0.03}(0.1) \\ \sqrt{0.06}(0.1) & 0.06 & 0.2 & \sqrt{0.03}(0.2+i0.05) & \sqrt{0.03}(0.3+i0.2) \\ \sqrt{0.045}(0.1) & (0.2-i0.2)\sqrt{0.03} & \sqrt{0.03}(0.1-i0.05) & 0.15 & (0.15)(0.2+i0.3) \\ \sqrt{0.045}(0.1) & (0.1)\sqrt{0.03} & (0.3-i0.3)\sqrt{0.03} & (0.15)(0.2-i0.3) & 0.15 \end{pmatrix}$$



y- axis :Log of Error, x-axis: No. of iterations. One can see the gradual increase in the error from -16.8 *when* $1 \leq x \leq 10$ *to* -15.7 *for* $980 \leq x \leq 1000$*. This shows that the truncation error is* 10^{-16} *and the total accumulated error is* $\sim 10^{-15}$ *which shows a pretty good performance for RK4.*

Figure: Error vs Iteration No.